

Unitarized ChPT Amplitudes and Crossing Symmetry Violation

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Abstract

Pion-pion scattering amplitude obtained from one-loop Chiral Perturbation Theory (ChPT) is crossing symmetric, however the corresponding partial-wave amplitudes do not respect exact unitarity relation. There are different approaches to get unitarized results from ChPT. Here we consider the inverse amplitude method (IAM) and, using the Roskies relations, we measure the amount of crossing symmetry violation when IAM is used in order to fit pion-pion phase-shifts to experimental data in the resonance region. We also show the unitarity violation of the crossing symmetric ChPT amplitude with its parameters fixed in order to fit to experimental phase-shifts.

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1 Introduction

Even though Quantum Chromodynamics (QCD) has achieved a great success in describing strong interactions, low energy hadron physics must still be modeled phenomenologically. A great theoretical improvement was made by means of the method of ChPT [1], which is an effective theory derived from the basis of QCD. The method consists of writing down chiral Lagrangians for the physical processes and uses the conventional technique of the field theory for the calculations.

Here we will focus on pion-pion scattering. For this process, the ChPT leading contribution (tree graphs) is of second order in the momenta p of the external pions and coincides with Weinberg result from current algebra [2]. The corrections come from loop diagrams whose vertices are of order p^2 and include a free-parameter polynomial part related to tree diagrams of order p^4 ; these parameters have to be obtained phenomenologically. At one-loop level the method yields a total amplitude that respects exact crossing symmetry, however, the corresponding partial-waves satisfy only approximate elastic unitarity.

This violation is more severe at higher energies, so that it is not possible to reproduce resonant states, which are one of the most relevant features of the strong interacting

regime. This is not a new issue in literature and many different methods have been proposed to improve this behaviour. Here we consider the inverse amplitude method (IAM) [3], that allows one to access the resonance region for pion-pion scattering by fixing two parameters, but violates crossing symmetry.

In the present exercise, our goal is to quantify this violation of crossing symmetry, what we do by calculating the deviations from the Roskies relations [4]. Our work is presented as follows. In section 2 we write the ChPT amplitude for pion-pion scattering and we construct IAM partial-waves. We introduce a correction to get rid of sub-threshold poles by slightly shifting the original Adler zeros of the leading amplitudes. In section 3 we display the so called Roskies relations, which follow from the requirement of exact crossing symmetry and involve integrals of the partial-wave amplitudes in the region $0 < s < 4m_\pi^2$. We measured the crossing symmetry violation of the IAM amplitudes. We obtained that some Roskies relations are violated at 30% level.

The compromise between crossing symmetry violation and approximate unitary amplitudes can be established by computing unitarity violation of the crossing symmetric ChPT amplitude, with parameters fixed in order to reproduce experimental phase-shifts. This is done in section 4 that also presents a summary of the main results.

2 Chiral perturbation theory and the IAM

In the case of pion-pion scattering, crossing symmetry implies that there is just one amplitude describing the three total isospin channels of the process. Using ChPT at the one-loop level and considering only the most relevant low energy constants, the amplitude can be decomposed as

$$A(s, t, u) = A^{ca}(s, t, u) + B(s, t, u) + C(s, t, u), \quad \text{where}$$

$$\begin{aligned} f_\pi^2 A^{ca}(s, t, u) &= s - m_\pi^2, \\ f_\pi^4 B(s, t, u) &= \frac{1}{6} \left(4 \left(s - \frac{1}{2} m_\pi^2 \right)^2 - \left(s - 2 m_\pi^2 \right)^2 \right) \bar{J}(s) \\ &\quad + \left[\frac{1}{12} \left(3 \left(t - 2 m_\pi^2 \right)^2 + (s - u) \left(t - 4 m_\pi^2 \right) \right) \bar{J}(t) + (t \leftrightarrow u) \right], \\ f_\pi^4 C(s, t, u) &= \lambda_1 \left(s - 2 m_\pi^2 \right)^2 + \lambda_2 \left[\left(t - 2 m_\pi^2 \right)^2 + (t \leftrightarrow u) \right]. \end{aligned}$$

The isospin defined amplitudes T_I for $I = 0, 1$ and 2 are

$$\begin{aligned} T_0(s, t) &= 3A(s, t, u) + A(t, s, u) + A(u, t, s), \\ T_1(s, t) &= A(t, s, u) - A(u, t, s), \\ T_2(s, t) &= A(t, s, u) + A(u, t, s), \end{aligned}$$

which are expanded in partial-wave amplitudes, as

$$T_I(s, t) = \sum_{\ell} (2\ell + 1) t_{\ell I}(s) P_{\ell}(\cos \theta),$$

where P_ℓ are the Legendre polynomials. In the following we omit the label ℓ , because we will just deal with S-wave ($I = 0, 2$) and P-wave ($I = 1$).

Elastic unitarity implies that, for $16m_\pi^2 \geq s \geq 4m_\pi^2$,

$$\text{Im } t_I(s) = \rho(s) |t_I(s)|^2,$$

which can be solved yielding

$$t_I(s) = \frac{1}{\rho(s)} e^{i\delta_I(s)} \sin \delta_I(s), \quad (1)$$

where $\delta_I(s)$ are the real phase-shifts and

$$\rho(s) = \frac{1}{16\pi} \sqrt{\frac{s - 4m_\pi^2}{s}}$$

is the phase space factor for pion-pion scattering. Even for the ChPT amplitude, that does not respect elastic unitarity constraint, the definition

$$\delta_I(s) = \arctan \frac{\text{Im } t_I(s)}{\text{Re } t_I(s)}, \quad (2)$$

will be used, in section 4.

At one-loop level, that is, up to order p^4 in the chiral expansion, the resulting ChPT amplitudes for isospin $I = 0, 2$ ($\ell = 0$, S-wave) and $I = 1$ ($\ell = 1$, P-wave) can be expanded as

$$t_I(s) = t_I^{ca}(s) + t_I^{ca\ 2}(s) \bar{J}(s) + t_I^{left}(s) + p_I(s), \quad (3)$$

where t_I^{ca} are the (real) Weinberg amplitudes, namely,

$$f_\pi^2 t_0^{ca}(s) = 2s - m_\pi^2, \quad f_\pi^2 t_1^{ca}(s) = \frac{1}{3}(s - m_\pi^2), \quad f_\pi^2 t_2^{ca}(s) = 2m_\pi^2 - s,$$

t_I^{left} are the parts that bear the left-hand cuts, namely,

$$\begin{aligned} f_\pi^4 \pi^2 t_0^{left}(s) &= \frac{1}{12} \frac{m_\pi^4}{s - 4m_\pi^2} (6s - 25m_\pi^2) L(s)^2 - \frac{1}{72\rho(s)} (7s^2 - 40m_\pi^2 s + 75m_\pi^4) L(s) \\ &+ \frac{1}{864} (95s^2 - 658m_\pi^2 s + 1454m_\pi^4), \\ f_\pi^4 \pi^2 t_2^{left}(s) &= \frac{-1}{12} \frac{m_\pi^4}{s - 4m_\pi^2} (3s + m_\pi^2) L(s)^2 - \frac{1}{144\rho(s)} (11s^2 - 32m_\pi^2 s + 6m_\pi^4) L(s) \\ &+ \frac{1}{1728} (157s^2 - 494m_\pi^2 s + 580m_\pi^4), \\ f_\pi^4 \pi^2 (s - 4m_\pi^2) t_1^{left}(s) &= \frac{1}{12} \frac{m_\pi^4}{s - 4m_\pi^2} (3s^2 - 13m_\pi^2 s - 6m_\pi^4) L(s)^2 \\ &+ \frac{1}{144\rho(s)} (s^3 - 16m_\pi^2 s^2 + 72m_\pi^4 s - 36m_\pi^6) L(s) \\ &- \frac{1}{864} (7s^3 - 71m_\pi^2 s^2 + 427m_\pi^4 s - 840m_\pi^6), \quad \text{with} \end{aligned}$$

$$\bar{J}(s) = \frac{1}{8\pi^2} - \frac{2}{\pi} \rho(s) L(s) + I \rho(s), \quad L(s) = \ln \frac{\sqrt{s - 4m_\pi^2} + \sqrt{s}}{2m_\pi},$$

and $p_I(s)$ are two free parameter polynomials, given by

$$\begin{aligned} f_\pi^4 p_0(s) &= \frac{1}{3} (11 s^2 - 40 s m_\pi^2 + 44 m_\pi^4) \lambda_1 + \frac{1}{3} (14 s^2 - 40 s m_\pi^2 + 56 m_\pi^4) \lambda_2, \\ f_\pi^4 p_1(s) &= \frac{1}{3} s (s - 4m_\pi^2) (\lambda_2 - \lambda_1), \\ f_\pi^4 p_2(s) &= \frac{2}{3} (s^2 - 2 s m_\pi^2 + 4 m_\pi^4) \lambda_1 + \frac{2}{3} (4 s^2 - 14 s m_\pi^2 + 16 m_\pi^4) \lambda_2. \end{aligned}$$

If one wants to describe a resonant amplitude, one may wish to use Padé approximants, as e.g. advocated in [3]. It amounts to writing the inverse of the partial-wave. Thus, instead of the exact ChPT result t_I , we use a modified amplitude

$$\tilde{t}_I(s) = \frac{t_I^{ca}(s)}{1 - (t_I^{ca})^2(s) \bar{J}(s) + t_I^{left}(s) + p_I(s)} / t_I^{ca}(s), \quad I = 0, 1 \text{ and } 2. \quad (4)$$

Our strategy was to choose the parameters λ_1 and λ_2 in order to fit S- and P-waves above to the experimental phase-shifts, by using the definition (1). We show in Fig. 1 the resulting phase-shifts corresponding to the parameters $\lambda_1 = -0.00345$ and $\lambda_2 = 0.01125$. As mentioned in the introduction, there was a problem concerning S-waves, namely that they were singular at some sub-threshold value for s , where the correction becomes equal to t^{ca} . Singularities occur in S-wave sub-threshold amplitudes at $s_0 \simeq 0.64 m_\pi^2$, for $I = 0$, and at $s_2 \simeq 1.95 m_\pi^2$, for $I = 2$. Those values are close to the ones where t_0^{ca} and t_2^{ca} actually vanish.

In order to get rid of those singularities, we performed an extra correction, thus obtaining a new partial-wave amplitude, denoted by $\tilde{t}_I^{(n)}$,

$$\tilde{t}_I^{(n)}(s) = \frac{(s - s_I)/f_\pi^2}{1 - (t_I^{ca})^2(s) \bar{J}(s) + t_I^{left}(s) + p_I(s)} / t_I^{ca}(s), \quad I = 0 \text{ and } 2. \quad (5)$$

The new formula slightly violates unitarity as can be measured by evaluating the quantity

$$Y^{IAM} = \frac{\text{Im } \tilde{t}_I^{(n)-1} - \rho}{\text{Im } \tilde{t}_I^{(n)-1} + \rho},$$

that is smaller than 2% in the energy range considered. One notices that the fits are not modified due to this correction, according to the phase-shift definition.

3 Crossing symmetry violation

As explained in the introduction, IAM allows one to access the resonance region for pion-pion scattering. On the other hand, the corresponding partial-waves do not respect

crossing symmetry and we would like to quantify that violation. Crossing symmetry imposes constraints between the integrals of some combinations of partial-wave amplitudes, known as Roskies relations [4]. Let us define

$$\begin{aligned}
A_1 &= 2 \int_0^{4m_\pi^2} f_0 ds, & B_1 &= 5 \int_0^{4m_\pi^2} f_2 ds, \\
A_2 &= \int_0^{4m_\pi^2} (3s - 4m_\pi^2) f_0 ds, & B_2 &= -2 \int_0^{4m_\pi^2} (3s - 4m_\pi^2) f_2 ds, \\
A_3 &= \int_0^{4m_\pi^2} (3s - 4m_\pi^2) f_0 ds, & B_3 &= 2 \int_0^{4m_\pi^2} f_1 ds, \\
A_4 &= \int_0^{4m_\pi^2} (3s - 4m_\pi^2) (2f_0 - 5f_2) ds, & B_4 &= 9 \int_0^{4m_\pi^2} f_1 ds, \\
A_5 &= \int_0^{4m_\pi^2} (10s^2 - 32sm_\pi^2 + 16m_\pi^4) (2f_0 - 5f_2) ds, & B_5 &= -6 \int_0^{4m_\pi^2} (5s - 4m_\pi^2) f_1 ds, \\
A_6 &= \int_0^{4m_\pi^2} (35s^3 - 180s^2m_\pi^2 + 240sm_\pi^4 - 64m_\pi^6) (2f_0 - 5f_2) ds, & & \\
& & B_6 &= 15 \int_0^{4m_\pi^2} (21s^2 - 48sm_\pi^2 + 16m_\pi^4) f_1 ds,
\end{aligned} \tag{6}$$

where $f_{0,2} = (s - 4m_\pi^2)t_{0,2}$ and $f_1 = (s - 4m_\pi^2)^2 t_1$. Crossing symmetric amplitudes must satisfy

$$A_i = B_i,$$

for i from 1 to 6. In order to quantify the amount of violation of these relations within the IAM, we evaluated the ratio

$$V_i = \frac{A_i - B_i}{A_i + B_i}. \tag{7}$$

We obtained the values shown in the Table.

4 Discussion and final remarks

The $\mathcal{O}(p^4)$ ChPT pion-pion amplitude is crossing symmetric but does not respect exact elastic unitarity. There are several attempts to extrapolate the domain of validity of ChPT and to access the resonance region for meson-meson scattering. One of these methods uses the inverse of the amplitude and fits the two-parameter amplitude to the experimental data.

In the present exercise we were interested in quantifying the crossing violation that this procedure implies. In order to do that we used the Roskies relations and we arrived to very big violations of crossing symmetry. On the other hand we have shown how to get rid of sub-threshold singularities by constructing a quasi-unitarized singularity corrected IAM amplitude.

For the sake of comparison, we show in Fig. 2 the fits of pure (crossing symmetric) ChPT amplitude to experimental phase-shifts, and evaluate, in turn, its unitarity violation. The fits were done as in Ref. [5], using definition (2). P-wave fit presents the same

i	1	2	3	4	5	6
IAM	8.94%	0.66%	1.37%	1.00%	29.2%	37.3%

Table 1: Percentage deviations on Roskies relations V_i , according to Eqs. 6 and 7.

quality as the corresponding IAM one, while $I = 0$ S-wave one is rather improved now. Here the parameters obtained are $\lambda_1 = 0.007520$ and $\lambda_2 = -0.00653$, which have the opposite signs in respect to those compatible with usual phenomenological ones. It happens because, in order to reproduce ρ -resonance, a complete inversion of the usual behaviour of the polynomial parts is required. It is an illustration of the current understanding of why it is not possible for pure ChPT amplitude to reach the resonance region.

It is thus interesting to evaluate the unitarity violation of that result. It is given, as before, by

$$Y^{UPCA} = \frac{\text{Im } t_I^{-1} - \rho}{\text{Im } t_I^{-1} + \rho}.$$

The results are shown in Fig. 3. We obtained values that reach more than 90%, for all three partial-waves. On the other hand, all Roskies relations vanish, within the numerical precision, of order 10^{-5} , in this case.

In summary our results show that it is not possible for ChPT to exactly fulfill all symmetry requirements, that is, by introducing elastic unitarity, a lot of crossing symmetry is lost, as well as keeping the latter costs a big amount of the former.

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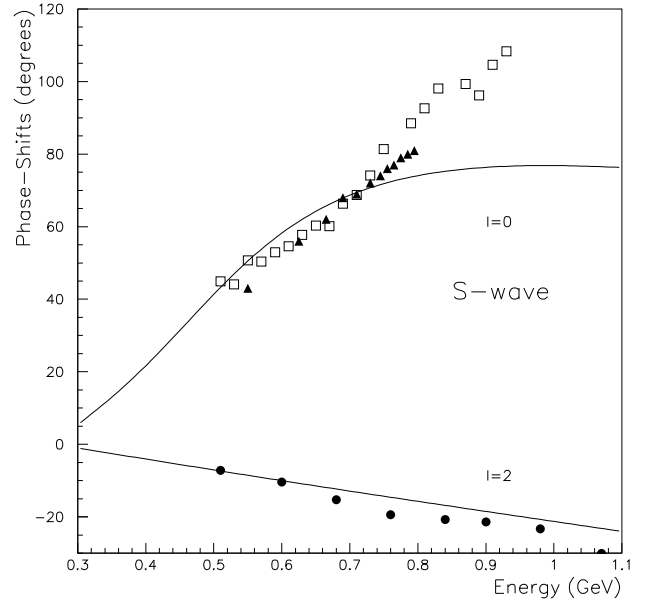
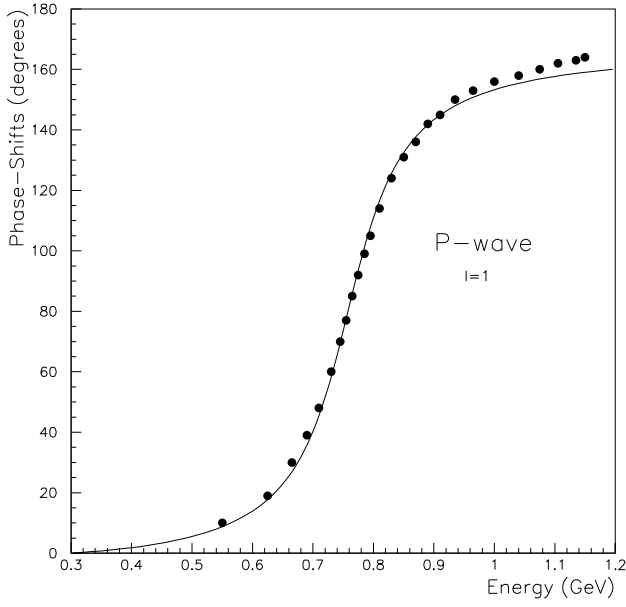


Figure 1: Results from fits of IAM amplitudes to P- and S-wave phase-shifts, in degrees, as functions of cms energy, in GeV. Experimental data for P-wave are from Ref. [6]; for S-wave, from Refs. [6, 7, 8].

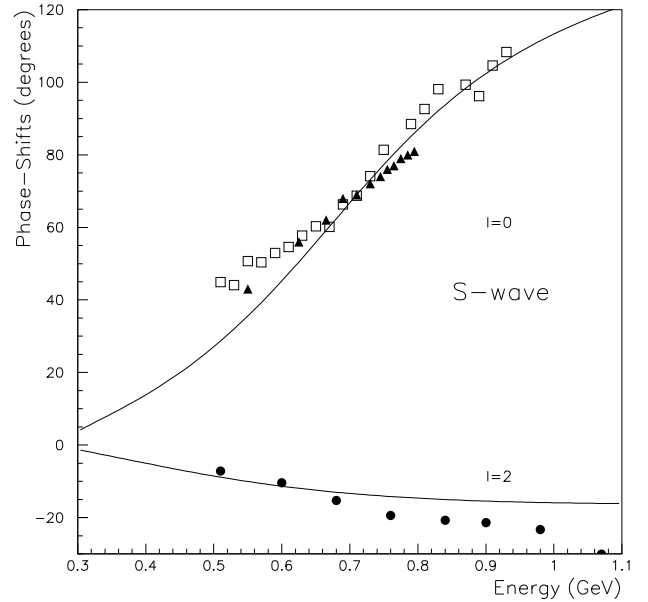
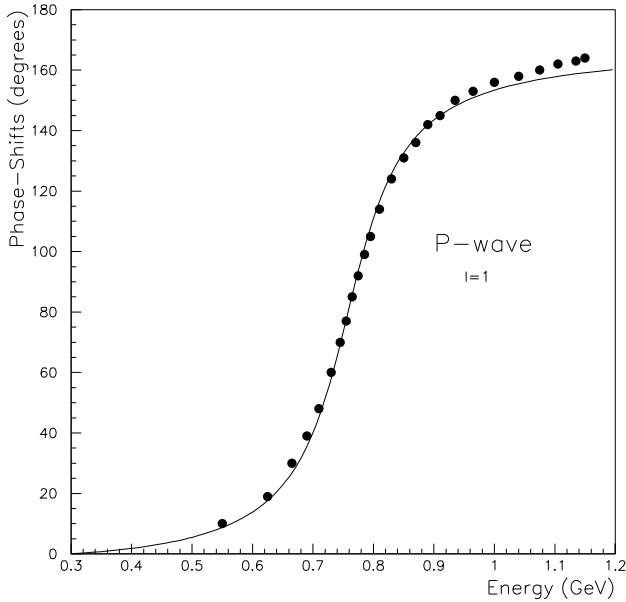


Figure 2: Results from fits of ChPT amplitudes to P- and S-wave phase-shifts, in degrees, as functions of cms energy, in GeV. Same experimental data as in Fig. 1.

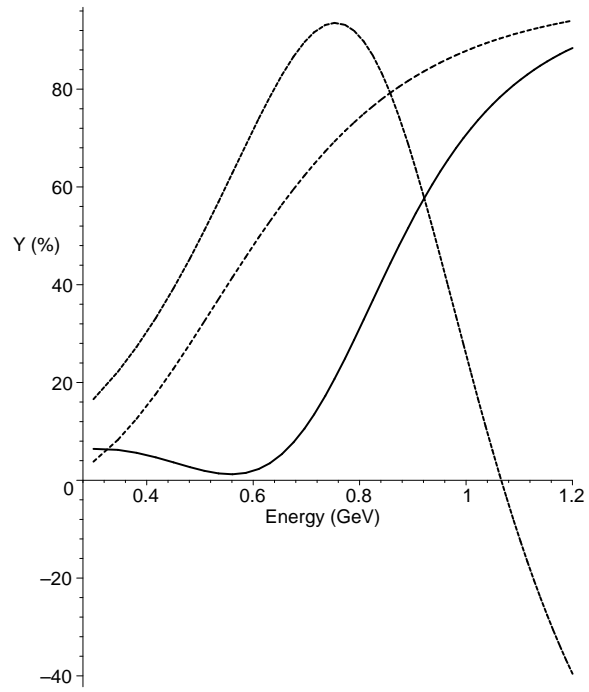


Figure 3: Unitarity violation (percentage) of ChPT amplitudes in P-wave (dotted), $I = 0$ (solid) and $I = 2$ (dashed) S-waves.